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On surface-wave radiation from a submerged cylindrical duct

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An integral-equation and variational formulation for the radiation of surface waves from a submerged cylindrical duct is developed. This formulation, which complements that of Simon (1981*a*) for the corresponding scattering problem, provides variational representations of the radiation impedance of the duct and (through the Green-function identities established by Simon) of the pressure-amplification factor in the scattering problem. An exact solution of the integral equation is obtained for the limiting case of a narrow duct.

1. Introduction

Simon (1981*a*) considers (i) the scattering of a surface gravity wave by, and (ii) the radiation from, a vertical circular duct of radius a with a submerged mouth at depth h in an ocean of infinite depth (figure 1). He develops an integral-equation formulation for, and obtains variational bounds to the amplitude of, the axisymmetric component of the scattered wave in problem (i) and calculates the pressure-amplification factor (the ratio of the wave-induced pressure in the depths of the duct to that at the depth h in the absence of the duct). He also determines the radiation damping and added mass in problem (ii) through indirect methods. I present here an integral-equation formulation of problem (ii), show that the radiation impedance in that problem and the magnitude of the pressure-amplification factor in problem (i) may be expressed in variational form, and obtain the exact solution for the limiting case of a narrow duct.

I follow Simon's notation except as noted. The velocity potential is given by

$$\operatorname{Re}\{(V/K)\,\phi e^{i\omega t}\}, \tag{1.1}$$

where V is the velocity in the depths of the duct,

$$K = \omega^2/g \tag{1.2}$$

is the wavenumber of the surface wave, ϕ is a dimensionless complex potential, and ω is the angular frequency ($V \equiv g\mathscr{A}Q/\pi a^2\omega$ and $\phi \equiv \pi K a^2 \Phi/Q$ in Simon's notation). The radiation problem then is prescribed by

$$\nabla^2\phi = 0, \tag{1.3}$$

$$\partial_r\phi = 0 \quad (r = a, z > h), \tag{1.4}$$

$$\phi \sim A H_0^{(2)}(Kr)\, e^{-Kz} \quad (Kr \to \infty), \tag{1.5}$$

$$(\partial_z + K)\,\phi = 0 \quad (z = 0), \tag{1.6}$$

$$\phi \sim \phi_0 - Kz \quad (r < a, z \to \infty), \tag{1.7}$$

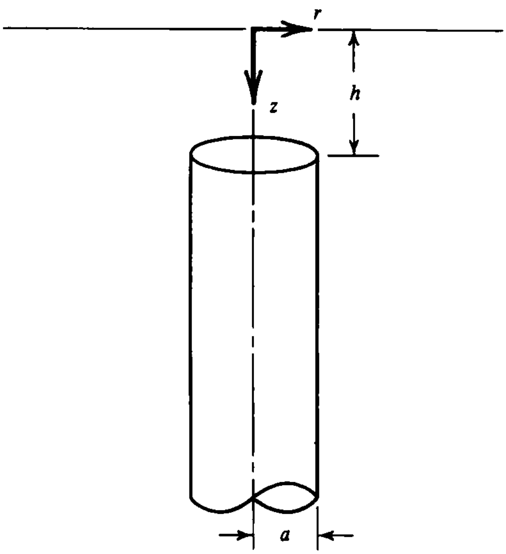

FIGURE 1. Submerged circular duct in an ocean of infinite depth.

where A and ϕ_0 ($A \equiv \pi K a^2 \hat{B}/Q$ and $\phi_0 \equiv Ka\hat{C}$ in Simon's notation) are complex constants that are to be determined.

The impedance of the duct, defined as the ratio of the complex amplitude of the perturbation pressure to that of the vertical velocity (positive up) as $z \to \infty$, is given by

$$\mathbf{Z} \equiv \lim_{z\to\infty}\left(\frac{-i\omega\rho\phi}{-\phi_z}\right) = i\rho\omega(z - K^{-1}\phi_0), \tag{1.8}$$

where ρ is the fluid density. The corresponding impedance for the column of water in $r < a$, regarded as moving independently of the water in $r > a$ but subject to the free-surface condition (1.6) so that $\phi = 1 - Kz$, is

$$\mathbf{Z}_< = i\rho\omega(z - K^{-1}), \tag{1.9}$$

wherein $i\rho\omega z$ represents the inertia of the column (the pressure required to force the oscillation of such a column is $\rho z i\omega V$), and $-i\rho\omega/K = \rho g/i\omega$ represents the gravitational restoring force at the free surface (at which the complex amplitude of the displacement is $V/i\omega$). The radiation impedance associated with the motion in $r > a$ is then given by

$$\mathbf{Z}_> \equiv \mathbf{Z} - \mathbf{Z}_< = i\rho c(1 - \phi_0), \tag{1.10}$$

where $c = \omega/K$ is the speed of the radiated wave. The real and imaginary parts of $\mathbf{Z}_>$, $\rho c \operatorname{Im}\phi_0$ and $\rho c(1 - \operatorname{Re}\phi_0)$, are measures of the radiation damping and the added mass (or stiffness if $\operatorname{Re}\phi_0 > 1$) of the water in $r > a$; $i\rho c$ is the impedance of a straight-crested gravity wave.

Simon applies Green's second theorem to the solutions of problems (i) and (ii) to obtain

$$A = \tfrac{1}{2}i\pi\mu^2 e^{-\tau}P, \quad |A|^2 = \tfrac{1}{2}\pi\mu^2 \operatorname{Im}\phi_0, \tag{1.11a, b}$$

where

$$\mu = Ka, \quad \tau = Kh, \tag{1.12a, b}$$

and P ($\equiv K_A$ in Simon's notation) is the complex pressure-amplification factor for the scattering problem.

2. Integral-equation formulation

Following Havelock (1929) and Simon (1981*a*), I develop the solution of (1.3) and (1.6) with the aid of the Fourier-transform pair (the present statement of this Fourier algorithm is somewhat more compact than those of Havelock and Simon)

$$F(k) = \int_0^\infty f(z)\, Z_k(z)\, dz \equiv \mathscr{F}\{f(z)\}, \tag{2.1a}$$

$$f(z) = 2KF(iK)\, Z_{iK}(z) + \frac{2}{\pi}\int_0^\infty F(k)\, Z_k(z)\,\frac{k^2 dk}{k^2+K^2} \equiv \mathscr{F}^{-1}\{F(k)\}, \tag{2.1b}$$

where ($Z_k \equiv k^{-1}M$ in Simon's notation)

$$Z_k(z) = \cos kz - Kk^{-1}\sin kz, \quad Z_{iK}(z) = e^{-Kz}. \tag{2.2a, b}$$

It follows from known results for Fourier integrals (Sneddon 1951) that

$$\lim_{z\to\infty} f(z) = -K^{-1}\lim_{k\to 0} k^2 F(k). \tag{2.3}$$

The solution of (1.3)–(1.7) may be expressed in terms of the dimensionless, complex amplitude of the radial velocity in the gap,†

$$\partial_r \phi|_{r=a} \equiv \tfrac{1}{2}Kaf(z) \quad (f \equiv 0 \quad \text{in} \quad z > h) \tag{2.4}$$

(the introduction of the scaling factor $\frac{1}{2}Ka$ simplifies the subsequent development), according to

$$\phi = \left.\begin{matrix} Z_0(z) + \frac{1}{2}Ka\mathscr{F}^{-1}\{[kI_0'(ka)]^{-1}F(k)\, I_0(kr)\} & (r < a) \\ \frac{1}{2}Ka\mathscr{F}^{-1}\{[kK_0'(ka)]^{-1}F(k)\, K_0(kr)\} & (r > a) \end{matrix}\right\}, \tag{2.5}$$

where $Z_0(z) = 1 - Kz$ ($k\to 0$ in (2.2*a*)), I_0 and K_0 are modified Bessel functions, the primes signify differentiation with respect to the argument, and, here and subsequently,

$$F(k) = \int_0^h f(z)\, Z_k(z)\, dz. \tag{2.6}$$

We remark that $\phi = Z_0(z)$, the first term on the right-hand side of (2.5), satisfies (1.3), (1.6) and (1.7) and would represent the fluid motion in the column $r < a$ if that motion were independent of the motion in $r > a$.

Requiring ϕ to be continuous across the gap ($r = a, 0 < z < h$) and invoking (2.1*b*) and the Wronskian relation for I_0 and K_0, we obtain the integral equation

$$\int_0^h G(z,\zeta) f(\zeta)\, d\zeta = Z_0(z) \quad (0 < z < h), \tag{2.7}$$

where

$$G(z,\zeta) = \frac{2Z_{iK}(z)\, Z_{iK}(\zeta)}{D(iKa)} - \frac{2K}{\pi}\int_0^\infty \frac{Z_k(z)\, Z_k(\zeta)\, dk}{(k^2+K^2)\, D(ka)} \tag{2.8}$$

and ($D(x) \equiv \bar{H}(x)/x$ in Simon's notation)

$$D(x) = 2I_1(x)\, K_1(x). \tag{2.9}$$

† An alternative formulation in terms of the potential discontinuity across the duct wall is possible, as in Simon's solution of the scattering problem, but does not appear to be worth pursuing here.

Letting $Kr \to \infty$ in (2.5), invoking (2.1*b*), comparing the result with (1.5), and invoking $K_0(ix) = -\frac{1}{2}i\pi H_0^{(2)}(x)$ and $H_0^{(2)\prime}(x) = -H_1^{(2)}(x)$, we obtain

$$A = -[Ka/H_1^{(2)}(Ka)]\,F(iK). \tag{2.10}$$

Letting $z \to \infty$ with $r < a$ in (2.5), invoking (2.3), and comparing the result with (1.7), we obtain

$$\phi_0 = 1 - F(0). \tag{2.11}$$

Combining (2.10) and (2.11) with (1.11), we obtain

$$P = 2i[\pi\mu H_1^{(2)}(\mu)]^{-1}\,e^{\tau}F(iK), \quad |P|^2 = 2\pi^{-1}\mu^{-2}e^{2\tau}\,\mathrm{Im}\,\phi_0 \tag{2.12a, b}$$

for the pressure-amplification factor, where μ and τ are defined by (1.12).

3. Variational formulation

Multiplying (2.7) through by $f(z)$, integrating over $(0, h)$, dividing the result through by $F^2(0)$, and invoking (2.11) and (1.10), we obtain the variational representation

$$\frac{\mathbf{Z}_>}{i\rho c} = 1 - \phi_0 = \frac{\left[\int_0^h f(z)\,Z_0(z)\,dz\right]^2}{\int_0^h\int_0^h f(z)\,G(z,\zeta)f(\zeta)\,d\zeta\,dz}, \tag{3.1}$$

for the ratio of the radiation impedance to that for a straight-crested gravity wave. Substituting G from (2.8) and invoking (2.6), we obtain the alternative form

$$1 - \phi_0 = \tfrac{1}{2}F^2(0)\left[\frac{F^2(iK)}{D(iKa)} - \frac{K}{\pi}\int_0^\infty \frac{F^2(k)\,dk}{(k^2+K^2)\,D(ka)}\right]^{-1}. \tag{3.2}$$

It follows from (3.2) that the real and imaginary parts of each of $\mathbf{Z}_>$ and ϕ_0 are stationary with respect to variations of $F(k)$ about the transform of the solution to (2.7); accordingly, $|P|^2$, as given by (2.12*b*), is also (but $\arg P$, as determined from (2.12*a*), is not) stationary with respect to such variations.

4. Rayleigh–Ritz approximation

Let $\{f_n(z)\}$ be a suitable set of functions, each of which satisfies the free-surface condition (1.6) at $z = 0$, is not more singular than $(h-z)^{-\frac{1}{2}}$ as $z \uparrow h$, and vanishes in $z > h$. Substituting the approximation

$$f(z) = \sum_{n=0}^{N} A_n f_n(z) \tag{4.1}$$

into (2.7), multiplying both sides of the result by $f_m(z)$, and integrating over $(0, h)$, we obtain

$$\sum_{n=0}^{N} G_{mn}A_n = \int_0^h f_m(z)\,Z_0(z)\,dz \equiv F_m(0) \quad (m = 0, 1, \ldots, N), \tag{4.2}$$

where

$$G_{mn} = \int_0^h\int_0^h f_m(z)\,G(z,\zeta)f_n(\zeta)\,d\zeta\,dz \tag{4.3a}$$

$$= \frac{2F_m(iK)\,F_n(iK)}{D(iKa)} - \frac{2K}{\pi}\int_0^\infty \frac{F_m(k)\,F_n(k)\,dk}{(k^2+K^2)\,D(ka)}, \tag{4.3b}$$

and $F_n(k)$ is derived from $f_n(z)$ through (2.6).

It is expedient, in choosing the f_n, to invoke the transformation (Simon 1981*a*)

$$\hat{f}_n(z) = f_n(z) - K \int_z^h f_n(\zeta)\, d\zeta, \tag{4.4}$$

which implies, through the above conditions on f_n,

$$\hat{f}_n'(z) = 0 \quad (z = 0), \quad \hat{f}_n(z) = 0 \quad (z > h), \tag{4.5a, b}$$

and, through (2.6),

$$F_n(k) = \int_0^h \hat{f}_n(z) \cos kz\, dz. \tag{4.6}$$

A complete set of expansion functions is then given by (Erdélyi *et al.* 1954)

$$F_n = J_{2n}(kh), \quad \hat{f}_n = (-)^n (2/\pi)(h^2 - z^2)^{-\frac{1}{2}} T_{2n}(z/h) \quad (0 < z < h), \tag{4.7a, b}$$

where the T_{2n} are Chebyshev polynomials. It follows from (4.7*a*) that $F_m(0) = \delta_{m0}$, whence the solution of (4.2) is given by

$$A_n = (-)^n M_{0n} |\mathbf{G}|^{-1}, \tag{4.8}$$

where $|\mathbf{G}|$ is the determinant of the $(N+1) \times (N+1)$ matrix $[G_{mn}]$ and M_{0n} is the corresponding minor of G_{0n}.

It follows from (4.1) and (4.7*a*) that $F(0) = A_0$, the substitution of which into (2.11) yields

$$\phi_0 = 1 - M_{00} |\mathbf{G}|^{-1}. \tag{4.9}$$

The equivalent of (4.9) also may be obtained by substituting

$$F(k) = \sum_{n=0}^{N} A_n J_{2n}(kh) \tag{4.10}$$

into (3.2) and requiring ϕ_0 to be stationary with respect to independent variations of each of the A_n. It follows from the variational principle that the error in the approximation (4.9) decreases monotonically with increasing N.

The real and imaginary parts of ϕ_0, calculated from (4.9) with $N = 2$, are plotted in figures 2 and 3. The corresponding approximations to $|P|$, $\arg P$ and $\hat{l} \equiv (\tau - \operatorname{Re}\phi_0)/\mu$ agree with those given by Simon's (1981*a*) figures 5, 7, 9 and 11 within the accuracy of the plots. A comparison among the results for $N = 0, 1, 2, 3$ reveals that the errors in the results for $N = 1/2$ are less than $1/0{\cdot}01\,\%$ for $0 < \mu < 2$, but that the results for $N = 0$ are good approximations only for $\mu \ll 1$. The present approximations are expected to be slightly more accurate than those of Simon in consequence of the failure of one of his three expansion functions to satisfy the counterpart of (4.5*a*) and perhaps also by virtue of the orthogonality of the present expansion functions.

An alternative solution of (2.7) through its reduction to a pair of real integral equations, one of which is equivalent to the integral equation for the scattering problem, is developed in the appendix.

5. The limit $\mu \downarrow 0$

Letting $\mu \downarrow 0$, which implies $D \to 1$, in (2.8) with z and ζ fixed and evaluating the integral over k with the aid of a table of Fourier integrals, we obtain

$$G(z, \zeta) \to Z_0 \binom{\zeta}{z} \quad (\zeta \lesseqgtr z, \mu \downarrow 0), \tag{5.1}$$

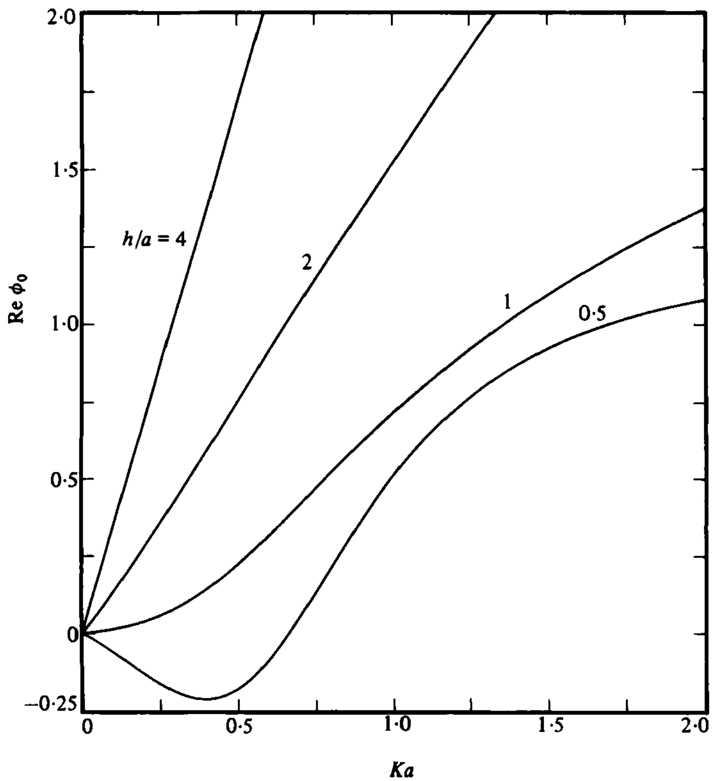

FIGURE 2. Re ϕ_0, as determined from (4.9).

where, here and subsequently, an error factor of $1+O(\mu^2 \log \mu)$ is implicit. The corresponding solution of the integral equation (2.7) is

$$f(z) = \delta(h-z), \tag{5.2}$$

where the delta function is defined such that

$$\int_0^h g(z)\,\delta(h-z)\,dz = g(h). \tag{5.3}$$

Note that, according to (5.2), the flow in the neighbourhood of the lip of the duct spreads out in a thin, annular sheet.

Substituting (5.2) into (2.6), which yields $F(k) = Z_k(h)$,† and invoking (2.12*a*), we obtain (the results (5.4*b*) and (5.5*b*) were obtained by Simon 1981*b*)

$$\phi_0 \to \tau, \quad P \to 1 \quad (\mu \downarrow 0). \tag{5.4a, b}$$

It then follows from (2.12*b*) that

$$\operatorname{Im} \phi_0 \to \tfrac{1}{2}\pi\mu^2 e^{-2\tau} \quad (\mu \downarrow 0). \tag{5.5a}$$

† The substitution of the limiting approximation $F(k) = Z_k(h)$ into the variational representation (3.2), which might be expected to yield a good approximation to ϕ_0, yields a divergent integral unless $D(ka)$ is approximated by its limiting value of 1. This difficulty is a consequence of the singular nature of the limit $\mu \downarrow 0$.

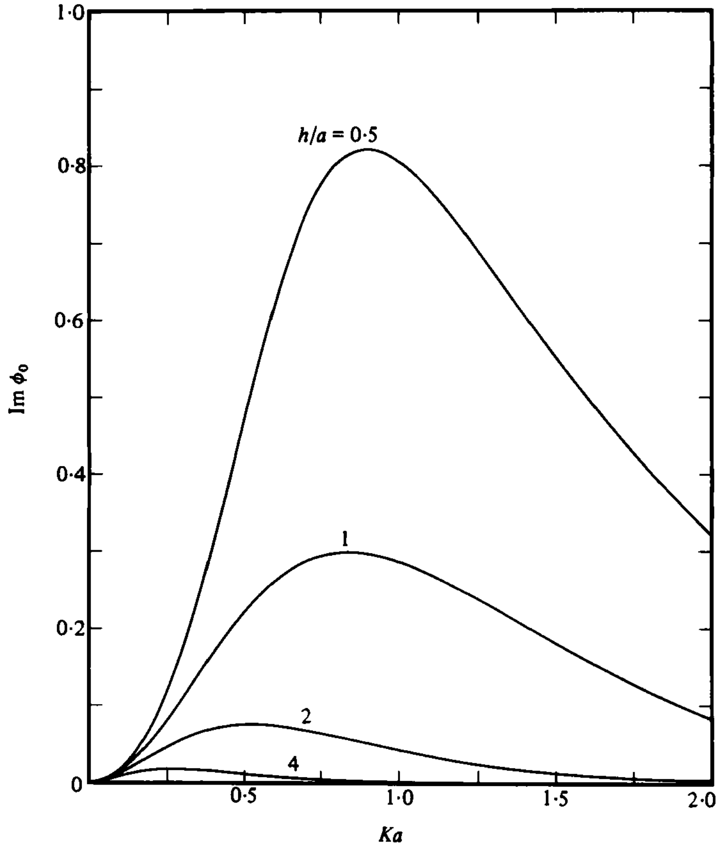

FIGURE 3. Im ϕ_0, as determined from (4.9).

The limiting approximation to Im P may be determined by perturbing (5.1) and (5.2) and is found to be

$$\operatorname{Im} P \to -\tfrac{1}{4}\pi\mu^2 e^{-2\tau} \quad (\mu \downarrow 0). \tag{5.5b}$$

The preceding limit is not uniformly valid as $\tau \downarrow 0$, in consequence of which ϕ_0 and P depend on h/a as $\mu \downarrow 0$ with h/a (rather than τ) fixed. In fact, (5.4*a*) is a rather good approximation for small but finite μ if h/a is sufficiently large (see figure 2, $h/a = 4$), but (5.4*a*) fails and $\operatorname{Re}\phi_0 < 0$ near $\mu = 0$ if h/a is sufficiently small (see figure 2, $h/a = 0{\cdot}5$). The limit (5.4*b*) remains valid for all h/a, but the approach to that limit depends on whether τ or h/a is fixed (cf. Simon's (1981*a*) figures 5 and 7).

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Appendix. Alternative solution of integral equation

The integral equation (2.7) may be recast in the form

$$\{2/D(i\mu)\}\, F(iK)\, e^{-Kz} + \int_0^h G_c(z,\zeta) f(\zeta)\, d\zeta = Z_0(z) \quad (0 < z < h), \tag{A 1}$$

where
$$G_c = -\frac{2K}{\pi}\int_0^\infty \frac{Z_k(z)\,Z_k(\zeta)\,dk}{(k^2+K^2)\,D(ka)} \tag{A 2}$$
is the contribution of the continuous spectrum to the kernel. Substituting
$$f(z) = f_a(z)+\{2/D(i\mu)\}\,F(iK)f_b(z), \tag{A 3}$$
where f_a and f_b are linearly independent functions, into (A 1) and separating the result, we obtain the real integral equations
$$\int_0^h G_c(z,\zeta)f_a(\zeta)\,d\zeta = Z_0(z), \quad \int_0^h G_c(z,\zeta)f_b(\zeta) = -e^{-Kz} \quad (0 < z < h). \tag{A 4a, b}$$
Taking the transform of (A 3), setting $k = iK$, and solving for $F(iK)$, we obtain
$$\{2/D(i\mu)\}\,F(iK) = \{\tfrac{1}{2}D(i\mu)-F_b\,iK)\}^{-1}F_a(iK). \tag{A 5}$$

Each of (A 4a, b) may be solved as in §4; in particular the solution for f_a is given by (4.1)–(4.8) if G is replaced by G_c in (4.3a) and the first term in (4.3b) is deleted. Moreover, (A 4b) is equivalent to equation (35a) of Simon (1981a), the integral equation for the scattering problem, from which it follows that
$$f_b(z) = \tfrac{1}{2}\pi\mu^{-1}f_S(z), \tag{A 6}$$
where f_S is Simon's f. It then follows from Simon's (35b) that
$$F_b(iK) = \tfrac{1}{2}\pi\mu^{-1}\zeta, \tag{A 7}$$
where ζ is Simon's scattering parameter, and from Simon's (48a) and (52) that
$$F_b(0) = \tfrac{1}{2}\pi\mu J_1(\mu)\,e^{-\tau}|P|\csc\alpha \quad (\alpha \equiv -\arg P). \tag{A 8}$$

Setting $k = 0$ in the transform of (A 3) and substituting $F(iK)$ from (2.12a), $F_b(0)$ from (A 8), and $D(i\mu) = -i\pi J_1(\mu)\,H_1^{(2)}(\mu)$ from (2.9), we obtain
$$F(0) = F_a(0)+\tfrac{1}{2}\pi\mu^2\,e^{-2\tau}|P|^2(\cot\alpha - i). \tag{A 9}$$
We remark that (A 9) is consistent with (2.11) and (2.12b).

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